COMPLEX DYNAMIC BEHAVIOR OF INVERTED PENDULUM WITH OSCILLATING SUSPENSION POINT: A REVIEW AND ANALYSIS

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Abstract. The study of inverted pendulum with vertically and horizontally oscillating suspension point is of interest to several fields of physics, mechanics and engineering. In this paper we begin with a short and comprehensive review of: 1) the theoretical pendulum and inverted pendulum investigations in Bulgaria and 2) some of the practical pendulum and inverted pendulum applications. In the second and third sections we investigate an inverted simple pendulum with oscillating suspension point by using Hamiltonian approach. The principal results are: critical curves (domain boundaries- bifurcation curves) met in the space of the horizontal and vertical Cartesian coordinates of the suspension point as two different behavior types (domains) of the averaged Hamiltonian system exist- stable and unstable. The fourth section is devoted to the numerical examples for a cycloidal (ordinary cycloid) oscillating suspension point. Regarding the case outlined above, the stability of the pendulum inverse state substantially depends on the amplitude of oscillations, i.e., initial conditions. Thus, the analytical and numerical results are in a good agreement.

Keywords: inverted pendulum, dynamics, stability, Hamiltonian approach

1. INTRODUCTION

The pendulum has a significant role in the development of human society, culture, education and science. The history of the physics of the pendulum started with experimental and theoretical studies of its practical applications [1, 2]. Perhaps, we might begin with the first modern scientist Galileo and

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its observation of the swinging chandeliers in the cathedral at Pisa. Galileo made the quantitative observation that the time/period of a swing, for a given pendulum, was independent of the amplitude of the pendulum's displacement. The pendulum had a central place in Newton's (classical) mechanics and it was studied in the works of Huygens, Hooke, Stokes, Atwood, Foucault and et al. Later, many engineering, chemical and biological processes as well as economic systems are modeled by using the pendulum motion.

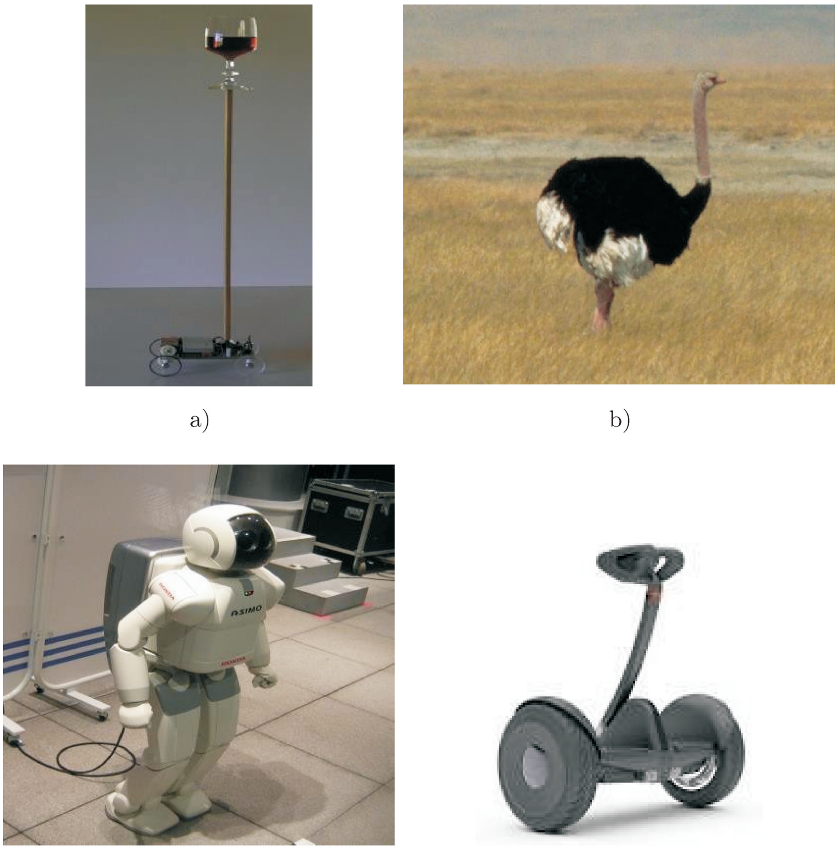
Scientific investigations of the pendulum motion (nonlinear oscillations) in Bulgaria began in 1939 with the theoretical work of professor Georgi Bradistilov – Corr. Member of the Bulgarian Academy of Sciences [3]. Professor Bradistilov was a prominent mathematician, engineer and patron of many Bulgarian researchers. The main objective of his pendulum papers is to investigate the stability and nonlinear behavior of composite pendulums [3-7]. The investigation of pendulum motion in Bulgaria entered into theoretical research with publications of S. Manolov [8], V. Dimova [9], St. Bachvarov [10], B. Cheshankov [11, 12], G. Boyadjiev [13], L. Karandzhulov [14] and many others.

More complex types of pendulums (such as the physical, mathematical, spring mass and torsional) can be used to demonstrate the oscillatory phenomena in techniques [15-21] and the movement of a child's swing [22–27].

The first theoretical studies (as dynamical system) on inverted pendulum started until the beginning of the twentieth century with the Stephenson's papers [28, 29]. There he demonstrated that inverted pendulum could be stabilized by applying rapid, vertical, harmonic oscillations to its base. Moreover, he obtained the stability conditions for the double and the triple inverted pendulums. Later, in 1932, the general equations for the inverted pendulums motion were developed by Lowenstern [30]. Note that the dynamics of inverted pendulums were well understood just in the 1960s [31].

The inverted pendulum played more than a scientific role in the formation and development of the modern engineering, technologies and industry [32]. It is widely used in research because the dynamics of many systems in robotics, human (biped locomotion), bicycle and segway is based on stabilization and balancing like to inverted pendulum [33–35] – see Fig. 1. Nowadays, it is well-known that the inverted pendulum is a classical nonlinear, under-actuated, inherently unstable and multivariable system [36–41] (see Fig. 2), which motion can be considered near its upper/lower position of equilibrium and the problem reduces to the analysis of the Mathieu equation by the Floquet theory (for linear periodic systems) [42, 43], and by the Ince-Strutt diagram [44, 45].

The dynamic stabilization of the inverted pendulum with a vertically and



a) b) c) d)

Fig. 1. Real examples of inverted pendulum: a) cart-pendulum system; b) biological (biped locomotion) systems; c) mechanical (robotic) models and d) segway (www.amazon.com)

horizontally oscillating suspension point (see Fig. 2) possesses an interesting behavior. It is well-known that, the inverted pendulum becomes stabilized after its instability, destabilizes again, and so forth ad infinitum [46]. Note that the first experimental example of this dynamic stabilization is the inverted pendulum with a vertically oscillating suspension point suggested by the Nobel prize-winner P. L. Kapitza [47]. It is well-known also that, when the vertically oscillations are increased, the inverted pendulum becomes stabilized one. On the other hand, when the horizontally oscillations are increased it becomes unstabilized [48]. Several investigators have sought to develop nonlinear dynamical models of the pendulum and the inverted pendulum when the suspension points oscillating one [4, 48–51]. Recently, the nonlinear chaotic dynamics of the inverted pendulums has attracted considerable attention [20, 52–55]. In [46] a parametrically forced pendulum with the vertically oscillating suspension point is considered. It is found that the inverted state stabilizes via alternating "reverse" subcritical pitchfork and period-doubling bifurcations, while it destabilizes via "normal" supercritical period-doubling and pitch-fork bifurcations. Unfortunately for an engineer, most of the theories developed are purely mathematical, which hampers their application to practical problems.

In the recent years, the research is focused towards obtaining control algorithms for general underactuated nonlinear mechanical systems. An underactuated mechanical system (UMS) is this which has fewer control inputs than degrees of freedom [41, 56]. UMSs have been arose in many different real-life applications such as: helicopters, aircrafts, spacecrafts, vertical take-off and landing aircrafts, under water vehicles, mobile robots, walking robots and so on. In some UMSs, the lack of action on certain directions can be interpreted as constrains on the acceleration. Some of these systems represent academic benchmarks and are part of a standard control laboratory like the inverted pendulum, the rotational inverted pendulum, the pendubot [57], the pendulum driven by a spinning wheel.

The investigation of dynamics of a simple inverted pendulum with oscillating suspension point is a classical problem of perturbation theory. According to [58–60], the problem can be simplified by using averaging in Hamiltonian form.

It is well-known that the systems called Hamiltonian if their equations can be written in canonical form by means of a Hamiltonian $H(q, p, t)$, where q and p are generalized coordinates and momenta, and t is the time. Consider the averaging (method) [61] in the Hamiltonian systems. Thus

$$H\left(q,p,\varepsilon\right) = H_0\left(p\right) + \varepsilon H_1\left(q,p,\varepsilon\right), \tag{1}$$

where $H_0\left(p\right)$ is the integrable unperturbed Hamiltonian, $H_1\left(q,p,\varepsilon\right)$ is the perturbed Hamiltonian, which is $2\pi-$periodic in components of q (fast variables), and ε is a small parameter (i.e. $\varepsilon \ll 1$) characterizing the magnitude of the perturbation. The equations of motion are given by

$$\begin{aligned} \dot{p} &= -\frac{\partial H(q,p,\varepsilon)}{\partial q} = -\varepsilon \frac{\partial H_1}{\partial q}, \\ \dot{q} &= \frac{\partial H(q,p,\varepsilon)}{\partial p} = \frac{\partial H_0(p)}{\partial p} + \varepsilon \frac{\partial H_1}{\partial p}. \end{aligned} \tag{2}$$

Using averaging method for the perturbed Hamiltonian H_1, we obtain the averaged Hamiltonian in the form

$$\widetilde{H}(p, \varepsilon) = H_0(p) + \varepsilon \widetilde{H}_1(p), \tag{3}$$

where $\widetilde{H}_1 = \dfrac{1}{(2\pi)^n} \displaystyle\int_{[0,2\pi)^n} H_1(q, p, 0) dq$. Here n is the number of degrees of freedom.

It follows from the Nekhoroshev's theorem [62, 63] that, if $H_0(p)$ is a step function, then there exist positive constants a, b and c such that for the perturbed Hamiltonian system (for a sufficiently small perturbation) we can suppose the form

$$|p(t) - p(0)| < \varepsilon^b, \qquad 0 \le t \le \left(1/\varepsilon\right) e^{\frac{c^{-1}}{\varepsilon^a}}. \tag{4}$$

The Hamiltonian system associated to $H_0(p)$ is integrable.

This paper is concerned to complete the study of the dynamical aspects of a mechanical system – simple inverted pendulum with oscillating suspension point.

In [50], we were interested in the stability of the inverted state when the suspension point makes vertically and horizontally oscillations, i.e., ordinary cycloidal oscillations; cardioidal oscillations and astroidal oscillations- for detailed information see [50] or Appendix. According to [48], using the Lagrange approach, i.e., the Euler–Lagrange equation

$$\frac{d}{dt}\frac{\partial L(t)}{\partial \dot{\theta}} - \frac{\partial L(t)}{\partial \theta} = 0, \tag{5}$$

the differential equation of motion of the inverted pendulum has the form

$$ml^2\ddot{\theta} + c\dot{\theta} - ml\left(g + \ddot{s}_1(t)\right)\sin\theta = ml\ddot{s}(t)\cos\theta, \tag{6}$$

where the Lagrangian L is defined as $L(\theta, \dot{\theta}, t)$. Notice that $\theta = \theta \mod 2\pi$ is the angle between the pendulum and the vertical (see Fig. 2), m is a mass attached to one end of a light rod (its mass can be negligible) of length l, c is a damping coefficient, $s(t)$ and $s_1(t)$ are the horizontally and vertically deviation of the suspension point A, respectively.

If we assume that the angle θ is small, (6) can be linerized by let $\sin\theta = \theta$, $\cos\theta = 1$, i.e.,

$$\ddot{\theta} = -\frac{c}{ml^2}\dot{\theta} + \frac{1}{l}(g + \ddot{s}_1(t))\theta + \frac{1}{l}\ddot{s}(t). \tag{7}$$

Notice that $s(t) = x_A$ and $s_1(t) = y_A$.

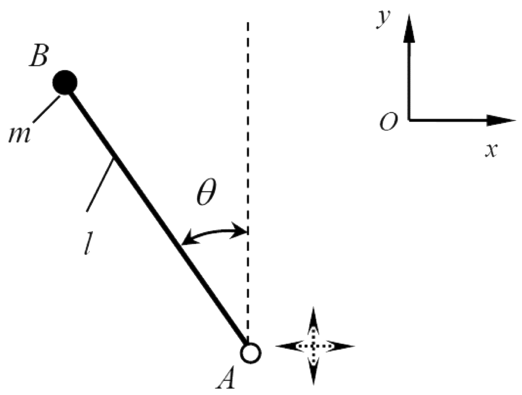


Fig. 2. Illustration of the inverted pendulum with the horizontally and vertically oscillating suspension point

Using (7), in [50] we obtained the analytical conditions for stabilization of the inverted state when the suspension point makes ordinary cycloidal oscillations, cardioidal oscillations and astroidal oscillations, respectively. Now, we continue our investigation by using Hamiltonian approach.

2. HAMILTONIAN OF THE INVERTED PENDULUM

Consider a simple inverted pendulum, Fig. 2, whose suspension point moves (oscillates) in the vertical and horizontal directions where the inverted pendulum moves. Let x_B, y_B be the horizontal and vertical Cartesian coordinates of the bob B, and let $x_A = s\,(t)\,,\ y_A = s_1(t)$ be the horizontal and vertical Cartesian coordinates of the suspension point A. Assume that of time t. Then we have x_A, y_A are some given functions of time t. Then we have

$$\left|\begin{array}{l} x_B = s\,(t) - l\sin\theta \\ y_B = s_1\,(t) + l\cos\theta \end{array}\right. \quad \text{or also} \quad \left|\begin{array}{l} \dot{x}_B = \dot{s}(t) - l\dot{\theta}\cos\theta \\ \dot{y}_B = \dot{s}_1(t) - l\dot{\theta}\sin\theta \end{array}\right. . \tag{8}$$

The kinetic energy T and the potential energy U of the bob have the form

$$\begin{array}{l} T = \dfrac{m}{2}\left(\dot{x}_B^2 + \dot{y}_B^2\right) = \dfrac{m}{2}\left[\dot{s}^2 + \dot{s}_1^2 + l^2\dot{\theta}^2 - 2l\dot{\theta}(\dot{s}\cos\theta + \dot{s}_1\sin\theta)\right] \\ U = mg\left(s_1 + l\cos\theta\right). \end{array} \tag{9}$$

To obtain the Hamiltonian $H(\theta,\ p, t)$ of the inverted pendulum, we pass to the canonical generalized momentum (conjugate to θ) representation

$$p = \frac{\partial T}{\partial \dot{\theta}}. \tag{10}$$

Thus, the Hamiltonian is

$$\begin{aligned} H &= p\dot{\theta} - L = p\dot{\theta} - T + U = \\ &= \frac{1}{2}\left[\frac{p^2}{ml^2} + \frac{2p}{l}(\dot{s}\cos\theta + \dot{s}_1\sin\theta) + m(\dot{s}\cos\theta + \dot{s}_1\sin\theta)^2)\right] + \\ &\quad + mgs_1 + mgl\cos\theta, \end{aligned} \tag{11}$$

where $L = T - U$ is the Lagrangian. Then there exist the governed canonical equations of motion in the form

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p} = \frac{1}{ml^2}p + \frac{1}{l}\left(\dot{s}\cos\theta + \dot{s}_1\sin\theta\right), \\ \dot{p} &= -\frac{\partial H}{\partial \theta} = \frac{1}{l}(\dot{s}\sin\theta - \dot{s}_1\cos\theta) + m(\dot{s}^2 - \dot{s}_1^2)\sin\theta\cos\theta + \\ &\quad + m\dot{s}\dot{s}_1\left(\sin^2\theta - \cos^2\theta\right) + mgl\sin\theta. \end{aligned} \tag{12}$$

For $H_1 = 0$ (see (1)), $H = H_0$, where the unperturbated Hamiltonian has the form

$$H_0(p) = \frac{1}{2ml^2}p^2. \tag{13}$$

In fact, the phase portrait of the system will be foliated by the curves of the Hamiltonian H_0. For $H_1 \neq 0$, the dynamics of the system (12) becomes more intricate.

3. AVERAGED HAMILTONIAN

Assume that $s = \varepsilon\bar{s}(t/\varepsilon)$ and $s_1 = \varepsilon\bar{s}_1(t/\varepsilon)$, where $\bar{s}$ and $\bar{s}_1$ are periodic functions with zero average. Then the averaged s_1 $(\langle s_1 \rangle)$ is equal to zero, and $\dot{s}$, $\dot{s}_1$ are from order 1 and can be neglected. According to [61], we average the Hamiltonian (11) with respect to time t. If $H_1 \neq 0$ and $0 < |\varepsilon| < 1$ then the averaged Hamiltonian $\widetilde{H}$ is

$$\begin{aligned} \widetilde{H} &= H_0 + \widetilde{U}(\theta) = \\ &= \frac{1}{2ml^2}p^2 + \frac{m}{2}\left[(K - N)\cos\theta + \Psi\sin 2\theta + K + N\right] + mgl\cos\theta, \end{aligned} \tag{14}$$

where

$$K = \frac{\dot{s}^2}{2}, \quad N = \frac{\dot{s}_1^2}{2}, \quad \Psi = \dot{s}\dot{s}_1. \tag{15}$$

Then, under the influence of the vibration in point A, a vibrated moment M_ν arises in the form

$$M_\nu = -\frac{\partial \widetilde{U}}{\partial \theta} = m\left[(K-N)\sin\theta - \Psi\cos 2\theta\right], \tag{16}$$

where $\widetilde{U} = -\min\limits_{\dot{\theta}} T$.

To an additive constant the averaged Hamiltonian $\widetilde{H}$ has the form

$$\widetilde{H} = H_0 + \widetilde{U}(\theta) = \frac{1}{2ml^2}p^2 + \frac{m}{2}\left[(K-N)\cos\theta + \Psi\sin 2\theta\right] + mgl\cos\theta. \tag{17}$$

It is essential to note that the type of the phase portrait of H depends on the extrema of $\widetilde{U}$. In fact, the function $\widetilde{U}$ effectively depends on two parameters $\frac{N-K}{gl}$ and $\frac{\Psi}{gl}$. If $\frac{\partial \widetilde{U}}{\partial \theta} = 0$ and $\frac{\partial^2 \widetilde{U}}{\partial \theta^2} = 0$, i.e.,

$$\begin{aligned} \frac{\partial \widetilde{U}}{\partial \theta} &= \frac{N-K}{gl}\sin 2\theta + \frac{\Psi}{gl}\cos 2\theta - \sin\theta = 0, \\ \frac{\partial^2 \widetilde{U}}{\partial \theta^2} &= \frac{2}{gl}\left[(N-K)\cos 2\theta - \Psi\sin 2\theta\right] - \cos\theta = 0, \end{aligned} \tag{18}$$

then the system (18) from two linear equations for unknown relations $\frac{N-K}{gl}$ and $\frac{\Psi}{gl}$ corresponds of a parametric curve with parameter θ. The effect of the upper equilibrium ($\theta = 0$) stabilization is a private example of stabilization for every pendulum position when both $\frac{\partial \widetilde{U}}{\partial \theta} = 0$ and $\frac{\partial^2 \widetilde{U}}{\partial \theta^2} = 0$ (see (18)) is a critical (bifurcation) curve corresponding to degenerate equilibria.

If $\frac{\partial^2 \widetilde{U}}{\partial \theta^2} > 0$, then the equilibrium ($\theta_0 = 0$) is stable (ellipsoid point – see Fig. 3b) – minimum of $\widetilde{U}$. Thus, we obtain the following condition for stability of upper equilibrium (inverted state)

$$\frac{|2(N-K)|}{gl} > 1. \tag{19}$$

As we have noted, if $\left.\frac{\partial \widetilde{U}}{\partial \theta}\right|_{\theta=\theta_0} = \left.\frac{\partial^2 \widetilde{U}}{\partial \theta^2}\right|_{\theta=\theta_0} = 0$ and $\left.\frac{\partial^3 \widetilde{U}}{\partial \theta^3}\right|_{\theta=\theta_0} \neq 0$ (or $\left.\frac{\partial^n \widetilde{U}}{\partial \theta^n}\right|_{\theta=\theta_0} \neq 0$, where n is an odd number), then $\theta_0 = 0$ is a point of inflexion. Here, we have

$$\left.\frac{\partial^3 \widetilde{U}}{\partial \theta^3}\right|_{\theta=\theta_0} = -\frac{4}{gl}\left[(N-K)\sin 2\theta + \Psi \cos 2\theta\right] + \sin\theta = 0. \tag{20}$$

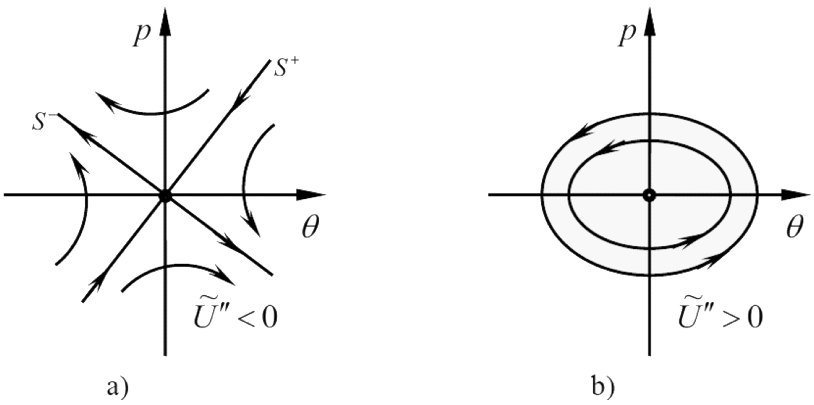


Fig. 3. Illustration of different type phase trajectories (points): (a) hyperbolic type point (trajectory) – unstable one and (b) ellipsoid type point (trajectory) – stable one. With S^+ and S^- are denoted the stable and unstable manifolds (separatrices)

Therefore, we have a degenerate function (curve), because $\left.\frac{\partial^n \widetilde{U}}{\partial \theta^n}\right|_{\theta=\theta_0} = 0$ – n is an odd number.

Solving system (18) for $\Omega = \frac{N-K}{gl}$ and $\Xi = \frac{\Psi}{gl}$, we obtain

$$\Omega = \frac{\cos\theta\cos 2\theta + 2\sin^2\theta}{1+\cos\theta - \cos 3\theta + \cos 4\theta}, \quad \Xi = -\frac{2\sin^3\theta}{1+\cos\theta-\cos 3\theta+\cos 4\theta}. \tag{21}$$

In Fig. 4 numerical solutions for Ω and Ξ (using (21)) are shown, when $\theta \in [-\pi/2, \pi/2]$. It is seen that Ω has values between 0.5 and 1, and Ξ is between -1 and 1. Notice that upper part of the curve is obtained for $\theta \in [-\pi/2, 0]$.

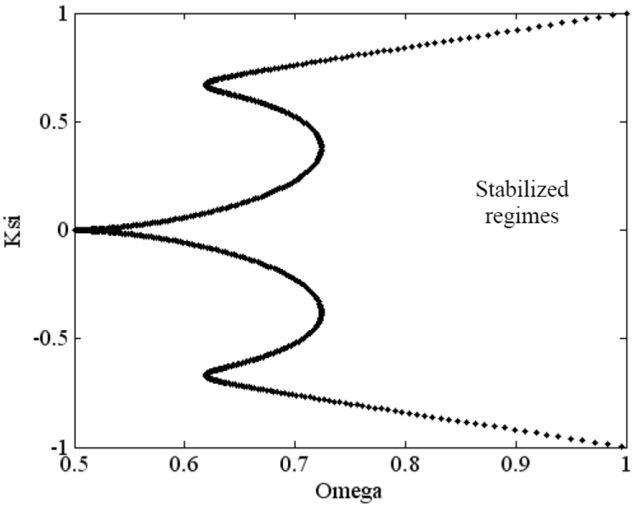


Fig. 4. Numerical solutions for Ω and Ξ, when $\theta \in [-\pi/2, \pi/2]$. With dependence on Ω and Ξ the borderline (bifurcation curve) between the stabilized and unstabilized regimes of inverted pendulum is shown

Since the function $\widetilde{U}$ is invariant under the transformation $\Psi \to -\Psi$, $\theta \to -\theta$, we can assume in our calculations that $\Psi \geq 0$.

Returning once more to the system (18) (or (21), respectively) from two linear equations for unknown relations Ω and Ξ, we note that two special cases 1) $\Xi = \Psi = 0$, 2) $\Omega = 0$, i.e. $K = N$ and a general case exist. The first special case (under an additional assumption that $\Omega = 0$) is considered by Stephenson, Bogolyubov, and Kapitza in [28, 29, 47, 61]. Below we consider the general case.

According to [65, 66], in the plane of parameters Ω and Ξ different behavior types (domains) of the averaged system exist, as the critical curves are domain boundaries – bifurcation curves. The critical curves can be two types: i) corresponding to degenerate equilibria – the first and the second derivatives of $\widetilde{U}$ vanish and ii) corresponding to saddle equilibria with equal values of potential energy – locally separate adjacent regions with different behavior of separatrices (Fig. 3a).

For the first type curves, its regular points correspond to centre-saddle bifurcations, i.e. centre and saddle equilibria coalesce and disappear [67–70]. Notice that these bifurcations are not degenerate. For the second type curves, the only possible type is the ray $\Psi = 0$, $\dfrac{2(N-K)}{gl} < 1$.

4. NUMERICAL EXAMPLES

In the previous section, we introduced the analytical tools that we shall use in our numerical analysis in the system (12). Basically, here we analyze numerically the dynamical model presented by the equations (12) for ordinary cycloidal oscillations of the suspension point, i.e. (A.1) takes place. The characteristic values of the moving inverted pendulum systems are: the mass attached to one end of the light rod (its mass can be negligible) m, length rod l, gravity acceleration g and frequency ω. In this work, the characteristic numerical values are [50]

$$m = 0.35 \text{ [kg]}, \quad l = 0.47 \text{ [m]}, \quad g = 9.81 \text{ [m/s}^2\text{]}, \quad \omega = 10 \text{ [s}^{-1}\text{]}. \tag{22}$$

Numerical calculations are made separately for two cases of oscillations of the suspension point and

$$\left| \begin{array}{l} s(t) = 0.05(10t - \sin 10t), \\ s_1(t) = 0.2(1 - \cos 10t), \end{array} \right. \tag{23}$$

$$\left| \begin{array}{l} s(t) = 0.47(10t - \sin 10t), \\ s_1(t) = 1(1 - \cos 10t), \end{array} \right. \tag{24}$$

For both cases, it is reasonable to accept (22) and define the initial conditions $\theta(0) = p(0) = 0$ for numerical calculations of the averaged system (12).

Keeping the inverted pendulum characteristic but changing the oscillations of the suspension point limits, we can demonstrate the inverted pendulum behaviors in different parts of the degenerate (bifurcation) curve in (Ξ, Ω) plane.

For (23) the points in partition of the parameter plane (Ξ, Ω) are plotted with black stars in Fig. 5a. All of them are in the zone, where the inverted state ($\theta = 0$) is unstabilized (the fixed point is hyperbolic one) and the system (12) becomes unstable (Figs. 5b, c). Notice that the condition (19) for stability of upper equilibrium (inverted state) is not valid. Thus, we have a good agreement between analytical and numerical results.

In Fig. 6, as the second case, the numerical results for system (12) are shown, when (24) is valid. In this case the inverted state is stabilized, as the fixed point $\theta = 0$ is from ellipsoid type. It is important to note that the condition (19) for stability of upper equilibrium is valid. Therefore, again we have a good agreement between analytical and numerical results.

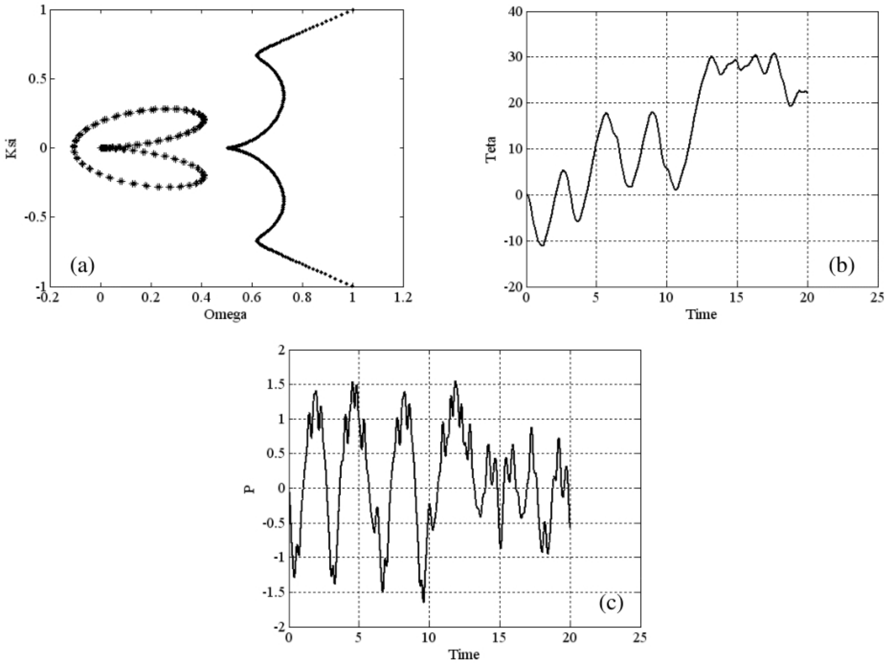


Fig. 5. Example of an unstable behavior of averaged system (12) using (23). (a) Borderline between regimes of stabilization and unstabilization of inverted state for the system (12) (dotted line), when for the suspension point (23) is valid (stars curve); (b), (c) unstable solution for q and p of the averaged system (12) using (23)

5. CONCLUSION

The pendulum is a marvelous physical system and is emblematic of a wide range of oscillating natural, technical and perhaps social systems. During the last 400 years, the pendulum played more than a scientific role in the development of technologies and modern industries. Historically, in the beginning of the twentieth century started the first findings around the dynamical behavior (stabilization) of inverted pendulum. Nowadays, the inverted pendulum plays an important role when it comes to systems control.

In this paper, firstly, we give a short and comprehensive review of the theoretical pendulum and inverted pendulum investigations in Bulgaria. After that, we consider and analyze some current aspects and problems in inverted pendulum applications. Secondly, we investigate the stabilization of inverted pendulum using Hamiltonian approach. Thus, we obtain critical curves which are domain boundaries – bifurcation curves. Basically, two different behavior

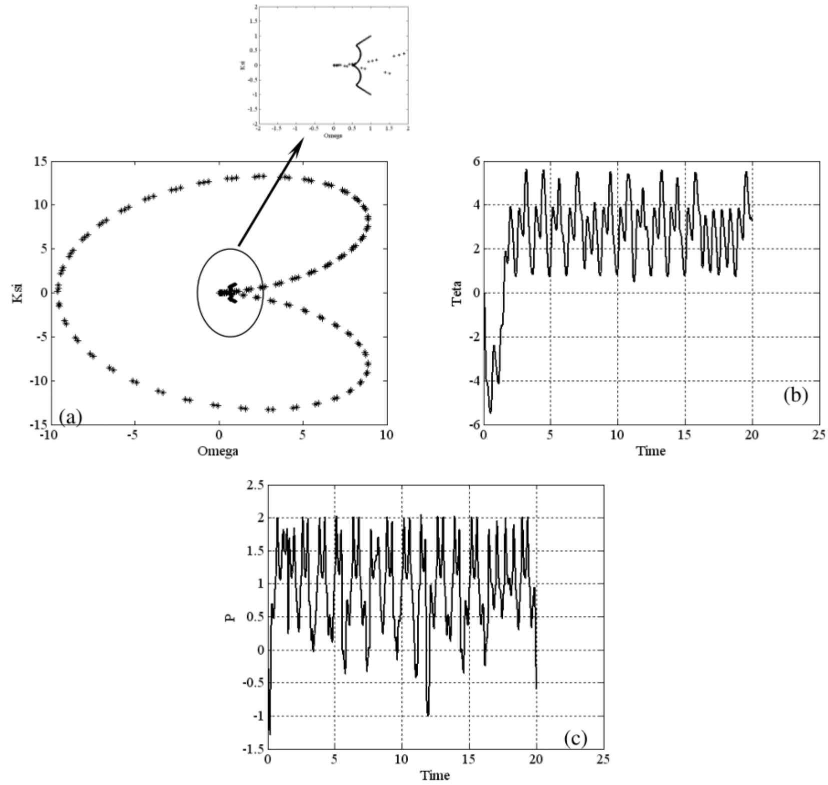


Fig. 6. Example of a stable behavior of averaged system (12) using (24). (a) Borderline between regimes of stabilization and unstabilization of inverted state for the system (12) (dotted line), when for the suspension point (24) is valid (stars curve); (b), (c) stable solution for q and p of the averaged system (12) using (24)

types (domains) of the averaged Hamiltonian system exist – stable and unstable. Notice that condition for stability of upper equilibrium (inverted state) was obtained.

Finally, numerical examples are shown, when the ordinary cycloidal oscillations of the suspension point take place. It is seen that analytical and numerical results are in a good agreement.

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REFERENCES

[1] G. Baker and J. Blackburn, *The pendulum: a case study in physics*, Oxford University Press, New York (2005), ISBN 0-19-856754-5.

[2] M. Matthews, C. Gauld and A. Stinner, *The pendulum: scientific, historical, philosophical and educational perspectives*, Springer, Dordrecht (2005), ISBN 10-1-4020-3525-X.

[3] G. Bradistilov, *Mathematische Annalen* (1939) **116** 181–203, ISSN 1432-1807.

[4] G. Bradistilov, *Prikl. Matematika i Mehanika* (1955) **19** (1) 113–118, ISSN 0032-8235 (in Russian).

[5] G. Bradistilov, *Bulletin de l'Institute de Matèmatiques* (1954) **1** (2) 135–145, ISSN 1310-6600.

[6] G. Bradistilov and G. Boyadjiev, *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik* (1959) **39** (7–8) 284–290, ISSN 1521-4001.

[7] G. Bradistilov, *Bulgarian Academy of Sciences, Bulletin de l'Institute de Matématiques* (1957) **2** (2) 73–85, ISSN 1310-6600.

[8] S. Manolov, *J. of Applied Mathematics and Mechanics* (1958) **22** (1) 139–142, ISSN 0021-8928.

[9] V. Dimova, *Mathématiques* (1965) **2** 132.

[10] St. Batsctwarow, *Ann. of VTUS Applied Mechanics* (1970) **4** (2) 135–143 (in Bulgarian).

[11] B. I. Cheshankov, *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik* (1974) **54** (5) 362–366, ISSN 1521-4001.

[12] B. I. Cheshankov, *Acta Mechanica* (1975) **21** (1–2) 41–52, ISSN 1619-6937.

[13] G. Boyadjiev, J. Dyn. Sys.*, Meas., Control.* (1989) **111** (4) 623–625, eISSN 1528-9028.

[14] L. I. Karandzhulov, *Ukr. Math. J.* (1991) **43** 710–719, ISSN 0041-6053.

[15] G. Bradistilov, G. Boyadjiev and A. Pisarev, *Ann. of MEI* (1964) **13** (1) 7–12 (In Bulgarian).

[16] St. Bachvarov, V. Zlatanov and S. Nikolov, Dynamics of a travelling crane with load during braking regime, X National Congress of Theoretical and Applied Mechanics, Varna, vol. 1, 27–33, 2005.

[17] H. Sheheitli and R. Rand, *Nonlinear Dynamics* (2012) **70** 25–41, ISSN 1573-269X.

[18] St. Bachvarov, V. Zlatanov and S. Atanasova, Dynamics of an elastic mathematical pendulum with a movable suspension point, XI National Congress on Theoretical and Applied, Mechanics, Borovets, vol. 3, 58–68, 2009.

[19] S. Bachvarov, V. Zlatanov and S. Atanasova, M*echanics, Transport, Communications* (2013) **11** (1) art. No 0770, ISSN 1312-3823.

[20] M. Gitterman, *The chaotic pendulum*, World Scientific, London (2010), ISBN-10 981-4322-00-8.

[21] S. Nikolov and V. Zlatanov, *Mechanics, Transport, Communications* (2019) **17** (3) art. No 1840, ISSN 1312-3823.

[22] W. Case and M. Swanson, *Amer. J. Physics* (1990) **58** 463–467, ISSN 1943-2909.

[23] S. Linge, *Comput. Methods Biomech. Biomed. Engin.* (2012) **15** 1103–1109, ISSN 1025-5842.

[24] S. Nikolov and D. Zaharieva, *Mechanics, Transport, Communications* (2017), **15** (3) art. ID 1495, ISSN 1312-3823.

[25] S. Nikolov, V. Vassilev and D. Zaharieva, Analysis of swing oscillatory motion, Studies in Computational Intelligence, in: Advanced Computing in Industrial Mathematics: 13th Annual Meeting of the Bulgarian Section of SIAM, December 18–20, 2018, Sofia, Bulgaria, Revised Selected Papers. Springer Nature, pp. 313–323 (2021), ISSN 1860-949X.

[26] S. Nikolov and V. Vassilev, *Journal of Geometry and Symmetry in Physics* (2023) **65** 93–108, ISSN 1312-5192.

[27] Ch. Hirata, SH. Kitahara, Y. Yamamoto, K. Gohara and M. Richardson, *Physical Review E* (2023) **107** (4) 044203, ISSN 2470-0053.

[28] A. Stephenson, *Memories and Proceeding of the Manchester Literary and Philosophical Society* (1908) **52** 1–10.

[29] A. Stephenson, *Philosophical Magazine* (1909), Series 6 **17** 765–766.

[30] E. Lowenstern, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* (1932) **13** (84) 458–486.

[31] K. Lundberg and T. Barton, *IFAC Proceedings Volumes* (2010), **42** (24) 131–135, ISSN 1474-6670.

[32] O. Boubaker and R. Iriate, *The inverted pendulum in control theory and robotics: from theory to new innovations*, UK, 137822, The Institution of Engineering and Technology, London (2017).

[33] N. Muskinja and B. Tovornik, *IEEE Transactions on Industrial Electronics* (2006) **53** (2) 631–639, ISSN 0278-0046.

[34] J. Furusho and M. Masubuchi, *Journal of Dynamic Systems, Measurement, and Control* (1987) **109** 155–161, ISSN 1528-9028.

[35] O. Druzhinina, L. Sevastianov, S. Vasilyev and D. Vasilyeva, *IOP Conf. Series: Journal of Physics: Conf. Series 937* (2017) 012011, ISSN 1742-6596.

[36] S. Nikolov and V. Nedev, *Mechanics, Transport, Communications* (2013) **11** (3), art No 0846 (7 pages), ISSN 1312-3823.

[37] S. Nikolov and V. Nedev, *Mechanics, Transport, Communications* (2013) **11** (2) art No 0775 (10 pages), ISSN 1312-3823.

[38] S. Nikolov and V. Nedev, *J. of Theoretical and Applied Mechanics* (2016) **46** (1) 17–32, ISSN 0861-6663.

[39] Ch. Chiu and YA. Peng, *Int. J. of Control, Automation and Systems* (2017) **15** 2343–2354, ISSN 1598-6446.

[40] K. Haddad, S. Belghith, H. Gritli and A. Chemori, *Int. J. of Bifurcation and Chaos* (2017) **27** (7), 1750104 (15 pages). ISSN 0218-1274.

[41] H. Gritli, N. Khraief, A. Chemori and S. Belghith, *Nonlinear Dynamics* (2017) **89** (3) 2195-2226, ISSN 1573-269X.

[42] L. Brillouin, *Wave Propagation in Periodic Structures*, 2end ed., Dover Publications, New York (1953).

[43] V. Yakubovich and V. Starzhinskii, *Linear Differential Equations with Periodic Coefficients*, John Wiley & Sons, Yerusalim-London (1975).

[44] E. Ince, *Mon. Notices R. Astron Soc.* (1915) **75** 436–448.

[45] E. Ince, *Proc. R. Soc. Edinb.* (1925–1926) **46** 20–29.

[46] S. Kim and Y. Kim, *Physical Review E* (2000) **61** (6) 6517–6520, ISSN 2470-0053.

[47] P. Kapitza, in Collected Papers of P. L. Kapitza (Ed. D. Ter Haar), Pergamon, London (1964), ISBN 9781483152677.

[48] T. Strijak, *Investigation Methods of the Dynamical Systems in the Pendulum Form*, Nauka, Alma-Ata, 1981 (in Russian).

[49] U. Neimark, N. Kogan and V. Saveliev, *Dynamical Models of Control Theory*, Nauka, Moscow, 1985 (in Russian).

[50] S. Nikolov, S. Bachvarov, *Eng. Mechanics* (2004) **11** (3) 201–214, ISSN 1802-1484.

[51] Z. Batalova, G. Beliakova and N. Buhalova, *Mehanika Tverdovo Telo-Izvestia Academii Nauk-USSR* (1987) **6** 18–25 (in Russian), ISSN 0572-3299.

[52] S. Kim, B. Hu, *Physical Review E* (1998) **58** (3) 3028–3035, ISSN 2470-0053.

[53] S. Kim, B. Hu, *Physical Review E* (1998) **58** (6) 7231–7242, ISSN 2470-0053.

[54] S. Nikolov, S. Bachvarov, V. Zlatanov and L. Klochkov, in: Proceedings of the 30-th Jubilee Int. Conference of TU Sofia, June 2004, Sozopol, Bulgaria, 2005, pp. 140–146.

[55] S. Nikolov, *Mechanics, Transport, Communications* (2012) **10** (1) art ID 650, ISSN 1312-3823.

[56] A. Braham, B. Cherki, M. Djemai and K. Busawon, *Analysis and Control of Underactuated Mechanical Systems*, Springer, New York (2014), ISBN 978-3-319-02635-0.

[57] M. Sprong, P. Corke and R. Lozano, *Automatica* (1996) **37** 1845-1851, ISSN 0005-1098.

[58] V. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York (1988), ISBN 978-1-4612-6994-6.

[59] V. I. Arnold, V. V. Kozlov and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*, Springer, Berlin (2006), ISBN 978-3-540-48926-9.

[60] A. Nayfeh and D. Mook, *Nonlinear Oscillations*, John Wiley & Sons, New York (1995), ISBN 047-112-142-8.

[61] N. Bogolyubov and Yu. Mitropol'skii, *Asymptotic Methods in the Theory of Non-linear Oscillations*, Gordon and Breach Science Publishers Ltd, New York (1961), ISBN 067-720-051X.

[62] N. Nekhoroshev, *Math. Sb. (N.S.)* (1973) **90**(**132**) (3) 432–478, ISSN 1064-5616.

[63] N. Nekhoroshev, *Uspekhi Mat. Nauk* (1977) **32**(**6**(198)) 5–66, ISSN 0036-0279.

[64] N. Beskin, *Analytic Geometry*, OGIZ-Gostehizdat, Moscow (1948) (in Russian).

[65] A. Neishtadt and K. Sheng, *Communications in Nonlinear Science and Numerical Simulation* (2017) **47** 71–80, ISSN 1878-7274.

[66] D. Treschev and O. Zubelevich, *Introduction to the Perturbation Theory of Hamiltonian Systems*, Springer-Verlag, Berlin Heidelberg (2010), ISBN 978-3-642-03027-7.

[67] J. Lowenstein, *Essentials of Hamiltonian Dynamics*, CUP, New York (2012), ISBN 978-1-107-00520-4.

[68] A. Pamuriño, *Electronic Journal of Qualitative Theory of Differential Equations, Monograph Series* (2005) **1** 1–204, ISSN 1417-3875.

[69] H. Hanssmann, *Local and Semi-local Bifurcations in Hamiltonian Systems*, Springer-Verlag, Berlin Heidelberg (2007), ISBN-10 3-540-38894-x.

[70] V. Sorokin, *Int. J. of Non-linear Mechanics* (2014) **63** 1–9, ISSN 0020-7462.

APPENDIX

A.1. Ordinary cycloidal oscillations of the suspension point

According to [64], the equation of the cycloid curve takes the form

$$\begin{aligned} s(t) &= \xi(\omega t - \sin\omega t), \\ s_1(t) &= \zeta(1 - \cos\omega t). \end{aligned} \tag{A.1}$$

If we differentiate twice the equation (A.1) and substitute into (6) we obtain [50]

$$\ddot{\theta} = -\frac{c}{ml^2}\dot{\theta} + \frac{1}{l}\left(g + \zeta\omega^2\cos\omega t\right)\theta + \frac{\xi\omega^2}{l}\sin\omega t. \tag{A.2}$$

A.2. Cardioidal oscillations of the suspension point
Here, the equation of the cardioid curve is

$$\begin{aligned} s(t) &= \xi(2\cos\omega t - \cos 2\omega t), \\ s_1(t) &= \zeta(2\sin\omega t - \sin 2\omega t). \end{aligned} \tag{A.3}$$

The second order derivative of the equation (A.3) can be substituted into (6) and as a result we obtain

$$\begin{aligned} \ddot{\theta} = &-\frac{c}{ml^2}\dot{\theta} + \frac{1}{l}\left(g + 8\zeta\omega^2 \sin\omega t \cos\omega t - 2\zeta\omega^2 \sin\omega t\right)\theta + \\ &+ \frac{1}{l}\left[4\xi\omega^2\left(\cos^2\omega t - \sin^2\omega t\right) - 2\xi\omega^2\cos\omega t\right]. \end{aligned} \tag{A.4}$$

A.3. Astroidal oscillations of the suspension point
The equation of the astroid curve has the form

$$\begin{aligned} s(t) &= \xi\cos^3\omega t = \frac{\xi}{4}(\cos 3\omega t + 3\cos\omega t\ ), \\ s_1(t) &= \zeta\sin^3\omega t = \frac{\zeta}{4}(3\sin\omega t - \sin 3\omega t\ ). \end{aligned} \tag{A.5}$$

In this case, after substitution of the second derivative of the equation (A.5) into (6), we have

$$\begin{aligned} \ddot{\theta} = &-\frac{c}{ml^2}\dot{\theta} + \frac{1}{l}\left(g + 6\zeta\omega^2 \sin\omega t \cos^2\omega t - 3\zeta\omega^2 \sin^3\omega t\right)\theta + \\ &+ \frac{1}{l}\left(6\xi\omega^2 \sin^2\omega t\cos\omega t - 3\xi\omega^2\cos^3\omega t\right). \end{aligned} \tag{A.6}$$

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