SIMPLE EQUATIONS METHOD (SESM): POWERFUL TOOL FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS OCCURRING IN NATURAL SCIENCES AND ENGINEERING

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Abstract. We describe the Simple Equations Method for obtaining exact solutions of nonlinear differential equations. A theorem is proved about the use of the equation $f_\xi = n\left[f^{(n-1)/n} - f^{(n+1)/n}\right], \xi = \alpha x + \beta t$ as a simple equation. Several examples are given for obtaining exact solutions of nonlinear partial differential equations by the use of SEsM with this simple equation. The examples include equations that are connected to the reaction-diffusion and reaction-telegraph equations. We show that SEsM is an effective methodology for solving nonlinear differential equations occurring in engineering.

Keywords: nonlinear differential equations, exact solutions, Simple Equations Method (SEsM), reaction-diffusion equation, reaction-telegraph-equation, tanh-method.

1. INTRODUCTION

Nonlinear partial differential equations (NPDE) are used a lot in the model- ing of natural systems [1]-[4]. Often, these equations possess traveling-wave solutions. Because of this, the traveling wave solutions of the nonlinear PDEs are studied very intensively [5]-[11]. Effective methods exist for obtaining exact traveling-wave solutions, e.g., the method of inverse scattering transform or the method of Hirota [12] - [14]. Many other approaches for obtaining exact special solutions of nonlinear PDEs have been developed in recent years (for example, see [15]). Let us note here only the method of the simplest equation and its version called the modified method of the simplest equation [16] - [21], as these methods are closely connected to the discussion below.

The method of the simplest equation is based on a procedure analogous to the first step of the test for the Painleve property [16] - [19]. In the version of the method called the modified method of the simplest equation [20] - [28], this procedure is

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substituted by the concept of the balance equations. We note that the balance procedure used in the modified method of the simplest equation may lead to more than one balance equation (such examples are presented below in the text). The modified method of the simplest equation has been successfully applied for obtaining exact traveling wave solutions of numerous nonlinear PDEs such as versions of the generalized Kuramoto - Sivashinsky equation, reaction - diffusion equation, reaction - telegraph equation [20], [22], the generalized Swift - Hohenberg equation and generalized Rayleigh equation [21], generalized Fisher equation, generalized Huxley equation [23], generalized Degasperis - Procesi equation and b-equation [24], extended Korteweg-de Vries equation [25], etc. [26] - [28]. Recently, the Modified method of Simplest Equation was further extended to the Simple Equations Method (SEsM) [29].

The text below is connected with applications of SEsM. The text is organized as follows. In Sect. 2, we briefly describe the methodology of SEsM and its relation to the Modified method of simplest equation. In Sect. 3, we present a theorem which is the main result of this study. Section 4 is devoted to several results connected to the application of the main result to several nonlinear differential equations. In Sect. 5, we present several concluding remarks and mention the opportunities for application of SEsM in the area of engineering sciences.

2. THE SIMPLEST EQUATIONS METHOD (SESM)

The SEsM is developed for obtaining exact and approximate solutions of systems of P nonlinear differential equations [30]-[34]. The solution is built as a composite function of exact solutions of Q simpler differential equations. Here, we describe a specific case $P = 1$ of SEsM (a solution of one nonlinear differential equation by solutions of M simpler equations). The Modified method of the simplest equation (MMSE) is a specific case of this. In MMSE, $Q = 1$.

SEsM has four steps. The solved differential equation is

$$D[u(x, \dots, t), \dots] = 0. \tag{1}$$

D depends on the function $u(x, \dots, t)$ and some of its derivatives. u can depend on several spatial coordinates. The idea of SEsM is to transform (1) to

$$\sum r_i(\dots)\mathcal{F}_i = 0. \tag{2}$$

$\mathcal{F}_i$ are some functions. r_i are the relationships among the parameters of the solved equation and the parameters of the solution. We set

$$r_i = 0, \tag{3}$$

and obtain a system of nonlinear algebraic equations. Each nontrivial solution of the system (3) leads to a solution of the solved equation (1).

The steps for the transition from (1) to (3) are as follows. Step 1 is a transformation of the nonlinearity of the equation to a more treatable kind of nonlinearity (for example, transformation to polynomial nonlinearity). An example of such a transformation is the Hopf–Cole transformation, which reduces the nonlinear Burgers equation to the linear heat equation [35, 36]. At Step 2 of the SEsM, one selects the form of the composite function for the solution of (1). One specific form for the case $n = 1$, $m = 1$ is the power series

$$F = \sum_{i=0}^{N} \mu_i f^i \text{ (μ is a parameter).}$$

At Step 3 of the SEsM, we determine the forms of the simpler equations and obtain one or more balance equations. The balance equations occur due to the requirement that any of r_i from (3) must have at least two terms in order to ensure that a nontrivial solution of the solved equation is obtained. In Step 4 of the SEsM, one solves the system (3). Each nontrivial solution of (3) leads to a nontrivial solution of the solved equation (1).

3. MAIN RESULT

Below, we shall consider traveling-wave-solutions $u(x,t) = u(\xi) = u(\alpha x + \beta t),$ constructed based on the simple equation.

$$f_\xi = n\,[\,f^{(n-1)/n} - f^{(n+1)/n}\,]\,, \tag{4}$$

where n is *an appropriate positive real number.* The solution of this equation is $f(\xi) = tanh^n(\xi)$ n must be such a real number that $tanh^n(\xi)$ exists for $\xi \in (-\infty, +\infty)(n = 1/3$ is an appropriate value for n and $n = 1/2$ is not an appropriate value for n).

Theorem. Let P be a polynomial of the function $u(x,t)$ and its derivatives $u(x,t)$ can be differentiated k times, where k is the highest order of derivative participating in P. P can contain some or all of the following parts:

(A) polynomial of u; (B) monomials that contain derivatives of u with respect to x and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of u; (C) monomials that contain derivatives of u with respect to t and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of u; (D) monomials that contain mixed derivatives of u with respect to x and t and/or products of such derivatives. Each such monomial can be multiplied by a polynomial of u; (E) monomials that contain products of derivatives of u with respect to x and derivatives of u with respect to t. Each such monomial can be multiplied by a polynomial of u; (F) monomials that contain products of derivatives of u with respect to x and mixed derivatives of u with respect to x and t. Each such monomial can be multiplied by a polynomial of u; (G) monomials that contain products of derivatives of u with respect to t and mixed derivatives of u with respect to x and t . Each such monomial can be multiplied by a polynomial of u; (H) monomials that contain products of derivatives of u with respect to x, derivatives of u with respect to t and mixed derivatives of u with respect to x and t. Each such monomial can be multiplied by a polynomial of u.

$$\mathrm{Ƥ} = 0 \tag{5}$$

We search for solutions to this equation of the kind $u(\xi) = \gamma + \delta f(\xi), \xi = \alpha x + \beta t$. γ and δ are parameters and $f(\xi)$ is a solution of the simple equation $f_\xi = n\,[f^{(n-1)/n} - f^{(n+1)/n}]$ where n is an appropriate real positive number. The substitution of this solution in Eq. (5) leads to a relationship R of the kind

$$R = \sum_{\sigma_i}^{S} K_{\sigma_i} f^{\sigma_i}, \tag{6}$$

where σ_i are some real numbers and K_{σ_i} are algebraic relationships containing the parameters of the solved equation and the parameters of the solution. Any nontrivial solution of the system of (nonlinear) algebraic equations $K_{\sigma_i} = 0$, $i = 1, \ldots$ leads to a solution of the solved nonlinear partial differential equation.

Proof. Let us denote the $k-$th derivative of $f(\xi)$ as $f_\xi^{(k)}$. First, we shall prove that if $f(\xi)$ obeys Eq. (4) then

$$f_\xi^{(k)} = \sum_{m=0}^{k} g_m(n) f^{(n-k+2m)/n}. \tag{7}$$

here $g_m(n)$ is a polynomial of n. In order to prove this, we mention that f_ξ satisfies this relationship as it can be seen from Eq. (4). The second, third, and fourth derivatives of $f(\xi)$

$$f_{\xi\xi} = n\left[(n-1)f^{(n-2)/n} - 2nf + (n+1)f^{(n+2)/2}\right],$$

$$f_{\xi\xi\xi} = n\left[(n^2 - 3n + 2)f^{(n-3)/n} + (-3n^2 + 3n - 2)F^{(n-1)/n} + \right.$$

$$\left.(3n^2 + 3n + 2)f^{(n+1)/n} - (n^2 + 3n + 2)f^{(n+3)/3}\right],$$

$$f_{\xi\xi\xi\xi} = n\left[-(n^3 - 6n^2 + 11n - 6)f^{(n-4)/n} - 4(n^3 - 3n^2 + 4n - 2)f^{(n-2)/n} + \right.$$

$$(6n^3 + 10n)f - 4(n^3 + 3n^2 + 4n + 2)f^{(n+2)/n} +$$

$$\left.(n^3 + 6n^2 + 11n + 6)f^{(n+4)/n}\right], \tag{8}$$

are of the kind (7). Let us assume that the k −th derivative of $f(\xi)$ is of kind (7). We shall show that the $(k+1)$-st derivative of $f(\xi)$ is of kind (7). From Eq.(7) we obtain $(q = k + 1)$

$$f_{\xi}^{(k+1)} = \sum_{m=0}^{q-1} g_m(n)(n - q + 2m + 1)f^{(n-q+2m)/n}$$
$$- \sum_{m=0}^{q-1} g_m(n)(n - q + 2m + 1)f^{(n-q+2m+2)/n}. \tag{9}$$

The term

$$\sum_{m=0}^{q-1} g_m(n)(n - q + 2m + 1)f^{(n-q+2m)/n}$$

is of kind (7). Let us split the term

$$\sum_{m=0}^{q-1} g_m(n)(n - q + 2m + 1)f^{(n-q+2m+2)/n}$$

in two parts:

$$\sum_{m=0}^{q-2} g_m(n)(n - q + 2m + 1)f^{(n-q+2m+2)/n}$$

and

$$\sum_{m=q-1}^{q-1} g_m(n)(n-q+2m+1)f^{(n-q+2m+2)/n}.$$

The first term above is of kind (7). Thus, for $f_\xi^{(k+1)}$ we have up to now all terms of the sum (7) except the last one (the term corresponding to $m = q$ that must contain $f^{(n+q)/n}$). But let us consider the second term above. This is exactly the missing term, as it is equal to $g_{q-1}(n)(n+q-1)f^{(n+q)/n}$ which is a term of the same kind as the k-th term of the sum in Eq.(7). Thus, the derivative $f_\xi^{(k+1)}$ is also of the kind (7), and our proposition is proven by the method of mathematical induction.

We have shown that $f_\xi^{(k)}$ is of the kind (6). Then it follows that any of the terms (A)-(H) is of the same kind or the kind f^0. Thus, P is reduced to a relationship of kind (6). Then any nontrivial solution of the system of nonlinear algebraic equations $\kappa_{\sigma i} = 0, i = 1,. . .$ leads to a solution of the solved nonlinear partial differential equation $\mathrm{P} = 0$.

4. EXAMPLES

As a first example, we shall consider the equation

$$u_t = p(u^a)x + q\left(u^b\right)_{xx}, \tag{10}$$

which is a kind of nonlinear reaction-diffusion equation [43]. We search for a solution of the kind $u(x,t) = \delta f(\xi), \xi = \alpha x + \beta t)$ where $f(\xi)$ is the solution of the simplest equation (4). The application of the balance procedure from the modified method of the simplest equation leads to the relationships $a = 1 + 2/n,\ b = 1 + 1/n$. Thus, Eq. (10) is reduced to the system of nonlinear algebraic equations

$$\beta - (n+1)q\alpha^2\delta^{1/n} = 0$$

$$p\delta^{1/n} - q\alpha(n+1) = 0$$

$$2q\alpha^2(n+1)\delta^{1/n} - \alpha p(n+2)\delta^{2/n} + 2n[q\alpha^2(n+1)\delta^{1/n} - \beta/2] = 0, \tag{11}$$

One solution of Eqs (11) is

$$\alpha = \frac{p}{(n+1)q}\delta^{1/n}; \qquad \beta = \frac{p^2}{(n+1)q}\delta^{3/n}, \tag{12}$$

and then the solution of the equation (10) $(a = 1 + 2/n,\ b\ = 1 + 1/n)$

$$u_t = p\left(1 + \frac{2}{n}\right)u^{2/n}u_x + q\frac{1}{n}\left(1 + \frac{1}{n}\right)u^{(1/n)-1}u_x^2 + q\left(1 + \frac{1}{n}\right)u^{1/n}u_{xx} \tag{13}$$

is

$$u(x,t) = \delta \tanh^n\left[\frac{p}{(n+1)q}\delta^{1/n}\left(x + p\delta^{2/n}t\right)\right] \tag{14}$$

Let us write several particular cases. For $n = 1$: the equation

$$u_t = 3pu^2u_x + 2qu_x^2 + 2quu_{xx}\ , \tag{15}$$

has the solution

$$u(x,t) = \delta \tanh\ \left[\frac{p}{2q}\delta\left(x + p\delta^2 t\right)\right]\ . \tag{16}$$

For $n = 2$: the equation

$$u_t = 2\text{pu}u_x + \frac{3q}{4}\ u^{-1/2}u_x^2 + \frac{3q}{2}u^{1/2}u_{xx}\ , \tag{17}$$

has the solution

$$u(x,t) = \delta \tanh^2\left[\frac{p}{3q}\delta^{1/2}\left(x + p\delta t\right)\right]\ . \tag{18}$$

For $n = 1/3$: the equation

$$u_t = 7pu^6u_x + 12qu^2u_x^2 + 4qu^3u_{xx}\ , \tag{19}$$

has the solution

$$u(x,t) = \delta \tanh^{1/3}\left[\frac{3p}{4q}\delta^3\left(x + p\delta^6 t\right)\right]\ . \tag{20}$$

For $n\ =\ 1/m$ (m is an odd integer): the equation

$$p(1 + 2m)u^{2m}u_x + qm(m+1)u^{m-1}u_x^2 + q(m+1)u^m u_{xx}\ , \tag{21}$$

has the solution

$$u(x,t) = \delta \tanh^{1/m}\left[\frac{mp}{(m+1)q}\delta^m\left(x + p\delta^{2m}t\right)\right] . \tag{22}$$

As a second example, we shall consider a nonlinear partial differential equation that is a generalization of the Maxwell-Cataneo kind of equation

$$u_t + ru^a u_{tt} = pu^{b-1}u_x^2 + qu^b u_{xx} . \tag{23}$$

The original Maxwell-Cataneo kind of equation is obtained when $a = 0$, $b = 1$ [44]. The use of the simplest equation (4) and the application of the methodology of the modified method of the simplest equation leads to two balance equations

$$a = b = \frac{1}{n}, \tag{24}$$

and the nonlinear partial differential equation (23)

$$u_t + ru^{1/n}u_{tt} = pu^{1/n-1}u_x^2 + qu^{1/n}u_{xx} , \tag{25}$$

is reduced to the system of nonlinear algebraic equations

$$r(n+1)\beta^2 - [(p\ +\ q)n + q]\alpha^2 = 0$$

$$\left\{r(n-1)\beta^2 - \left[\left((p+q)n - q\right)\right]\alpha^2\right\}\delta^{1/n} + \beta = 0$$

$$\left[(p+q)\alpha^2 - r\beta^2\right]n\delta^{1/n} - \frac{1}{2}\beta = 0 . \tag{26}$$

One non-trivial solution of this system is

$$\alpha = \frac{1}{2npr^{1/2}\delta^{1/n}}\left[(n+1)(np+nq+q)\right]^{1/2} \ ; \ \ \beta = \frac{np+nq+q}{2npr\delta^{1/n}} \ , \tag{27}$$

and the corresponding solution of Eq.(25) is

$$u(x,t) = \delta \tanh^n \left\{ \frac{1}{2npr^{1/2}\delta^{1/n}} \left[\left((n+1)(np+nq+q) \right)^{1/2} x + \frac{np+nq+q}{r^{1/2}} t \right] \right\}. \tag{28}$$

Several particular cases are as follows. For $n = 1$: the equation

$$u_t + ruu_{tt} = pu_x^2 + quu_{xx} , \tag{29}$$

has the solution

$$u(x,t) = \delta \tanh \left\{ \frac{1}{2pr^{1/2}\delta} \left[\left(2(p+2q) \right)^{1/2} x + \frac{p+2q}{r^{1/2}} t \right] \right\}. \tag{30}$$

For $n = 2$: the equation

$$u_t + ru^{1/2}u_{tt} = pu^{-1/2}u_x^2 + qu^{1/2}u_{xx} , \tag{31}$$

has the solution

$$u(x,t) = \delta \tanh^2 \left\{ \frac{1}{4pr^{1/2}\delta^{1/2}} \left[\left(3(2p+3q) \right)^{1/2} x + \frac{2p+3q}{r^{1/2}} t \right] \right\}. \tag{32}$$

Finally, let $n = 1/3$. Then the equation

$$u_t + ru^3u_{tt} = pu^2u_x^2 + qu^3u_{xx} , \tag{33}$$

has the solution

$$u(x,t) = \delta \tanh^{1/3} \left\{ \frac{3}{2pr^{1/2}\delta^3} \left[\left(\frac{4}{9}(p+4q) \right)^{1/2} x + \frac{p+4q}{3r^{1/2}} t \right] \right\}. \tag{34}$$

The last example is connected to the following equation from the class of reaction-telegraph equations [45]:

$$u_t - au_{tt} - bu_{xx} - c(q+2ru)u_t - (p+qu+ru^2) = 0. \tag{35}$$

The application of the methodology of the modified method of the simplest equation on the basis of a solution $u(x,t) = \sigma + \delta f(\xi)$, $\xi = \alpha x + \beta t$ where $f(\xi)$ is solution of the simple equation (4) leads to the balance equation $n = 1$ and to the reduction of Eq.(35) to the following system of nonli near partial differential equations

$$a\beta^2 + \alpha^2 b - \beta\delta c r = 0,$$

$$\beta(2\sigma c r + c q - 1) - \delta r = 0,$$

$$\beta\delta c r - a\beta^2 + \alpha^2 b - \sigma r - \frac{1}{2} q = 0$$

$$\beta\delta(1 - 2cr\sigma - cq) - r\sigma^2 - q\sigma - p = 0. \tag{36}$$

One nontrivial solution of the system of equations (36) is

$$\alpha = -\frac{1}{2b}[b(a+c)(4pr - q^2)]^{1/2}; \quad \beta = -\frac{1}{2}(q^2 - 4pr)^{1/2}$$

$$\sigma = -\frac{q}{2r}; \;\; \delta = \frac{1}{2r}(q^2 - 4pr)^{1/2}; \tag{37}$$

and the corresponding solution of Eq.(35) is

$$u(x,t) = -\frac{q}{2r} - \frac{1}{2r}(q^2 - 4pr)^{1/2} \tanh\left\{\frac{1}{2b}[b(a+c)(4pr - q^2)]^{1/2} x + \frac{1}{2}(q^2 - 4pr)^{1/2} t\right\} \tag{38}$$

5. DISCUSSION AND CONCLUDING REMARKS

In this article, we discuss the application of version SesM (P, Q) ($P = 1, Q = 1$) of the Simple Equations Method (SEsM) for obtaining exact and approximate solutions to nonlinear differential equations. We enriched the methodology by adding to it a new simple equation of kind (4), whose solution is $\tanh^n(\xi)$, $\xi = \alpha x + \beta t$. n can be an appropriate positive real number. We discuss several examples for the application of the methodology for obtaining exact solutions of kink and solitary wave kinds to nonlinear partial differential equations that are used to model reaction-diffusion and reaction-telegraph systems. As it can be seen from the text, the much-used tanh-method for obtaining exact solutions of nonlinear partial differential equations is a particular case of SEsM (1,1), and this particular case is realized when the simple equation is (4) with $P = 1$.

SEsM methodology is very convenient for obtaining exact solutions of nonlinear differential equations possessing polynomial nonlinearities. Such equations occur frequently in the area of engineering. The cases of weak nonlinearities are widely studied there. In these cases, one presents the nonlinear terms as a Taylor series. Thus, polynomial nonlinearities occur. These nonlinearities can be treated by SEsM even without using Step. 1 of the methodology. Thus, the methodology may have many applications to engineering problems. Some of these applications will be shown in our future research.

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